THE

PHILOSOPHER'S STONE

BY

P. H. VANDER WEYDE, M. D.

FOUR ESSAYS

CONTAINING THE ANSWER OF

POSITIVE SCIENCE,

TO THE QUESTION: WHAT IS KNOWN AT PRESENT

ABOUT

1. THE QUADRATURE OF THE CIRCLE.
2. PERPETUAL MOTION.
3. THE MAKING OF GOLD.
4. THE ELIXIR OF LIFE.

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Preface.

These four Essays treat on four branches of Positive Science in the natural progressive order in which they ought to be studied; the first, on *The Quadrature of the Circle* relates to MATHEMATICS, the Science of the *laws of Space and of Time*; the second on *Perpetual Motion*, to MECHANICS, or *the laws of Motion or Force*; the third on the *Making of Gold*, to CHEMISTRY or the study of the *laws of the Atomic Affinities*; and the fourth on the *Elixer of Life*, relates to BIOLOGY, the science of the *Laws Governing Life*.

The purpose of this publication is twofold:

First, To demonstrate that the research after these subjects belongs to bygone ages, when science was in its infancy: so that no person acquainted with the modern state of the sciences to which they relate, will waste his time with such an unfruitful occupation.

Second, To show that, if those sciences were more generally studied in the order given above, we would not only have no more researches after the philosopher's stone, etc. but have also less obscure notions introduced in the branches belonging to BIOLOGY, the science of *the laws of Life*.

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We find, for instance, that the preparation for the profession of Physician is, at present, in this country, conducted in such a way, that a M. D. usually has no knowledge whatever of *Mathematics* or *Mechanics*, and only a very limited knowledge of *Natural Philosophy and Chemistry*. Therefore it is no wonder that by this defect in the preparatory education of many authors and teachers of Medical Science, in the two branches of *Biology*—*Physiology*, and *Pathology*—empty words are still used in place of thorough explanations, and are sustained, even defended, against the discoveries of modern chemistry, by those who, ignorant of those discoveries, will, of course for their own convenience sake, use such words as *Vitality*, *Nervous Action*, *Healing Power of Nature*, etc. The use of such expressions is the easiest way to cut short all farther enquiry, and their use therefore is very natural, because a person ignorant of all Mathematical, Mechanical, and Chemical Laws is of course unable to explain phenomena depending on those Laws.

The *first* essay, that on the *Quadrature of the Circle*, was written long ago, and is a translation of an article which the author prepared for the Journal of Mathematics and Physics edited by him in Holland, in 1840 – 1846.

The *second* essay, that on *Perpetual Motion*, contains many quite new ideas. This subject having lately received such immense light by the discovery of the Unit of Heat by JOULE, and still later in its general connection with thermology, has originated the new branch of *Thermo-dynamics*, and established the fact that Caloric is the base and the primary cause of all Force, a principle at present known under the name of "the doctrine of the correlation of forces."

The *third* essay, that on Chemistry, is in its nature more historical, and relates to the science of Chemical discoveries and to the probability of the transmutation of elementary matter.

Finally the *fourth* essay, that on Biology, will contain a condensed exposition of human physiology and pathology, and demonstrate how this science, considered pure and theoretical, is only in its infancy compared with the preceding, notwithstanding the immense mass of experience handed down from HIPPOCRATE's time to the present day.

Cooper Institute Laboratory,
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P. H. VAN DER WEYDE, M. D.

No. 1.

THE QUADRATURE

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There exists no Geometrical Problem more celebrated and popular than that of *the Quadrature of the Circle*; the innummerable attempts to which it gave rise, the follies committed in those attempts, the exagerated interest attached to it, all contributes to make its history very interesting. However it is not our intention to write this History; already a century ago it filled some large volumes, and would occupy more space now; we will rather confine ourselves to the more useful task of exposing the results of the most valuable researches on this subject, especially those of modern times.

The *Quadrature of the Circle* is: *The finding of the Side of a Square equal to a given Circle.* This Problem would be solved if we could construct *Geometrically* (that is with Rule and Compass only) a straight line equal to the circumference of a given circle, because Geometry demonstrates that the surface of a Circle is equal to a Triangle having for its Base the Circumference and for its Height the Radius. As the side of a Square of equal surface to that of a triangle is mean proportional between the Height and half the Base of that Triangle, the problem is reduced to finding the Circumference for a given Radius; or, in general, *to finding the Proportion between the Diameter and Circumference*, this Proportion being the same for all Circles.

The investigation of this proportion belonging to transcendental Mathematics, as we will see later, common geometry can only solve the problem by approximation. We will now explain the way in which it is done, as unfortunately most elementary works on Geometry contain nothing about this beautiful and simple operation, giving to the student the number representing the proportion as a matter of belief, which is entirely contrary to the spirit and purpose of mathematical study, and, besides has the injurious effect of unsettling many minds, one of which now and then wastes his time in an unprofitable research, more frequently than is supposed. The result is that in Germany, France, Holland, England, the United States, and even Italy and Spain, every ten or twelve years a book appears in print, written by a Mr. Smith, Jones, Parker, et. al., pretending to have solved the great problem of the quadrature of the circle, a problem in fact solved long ago.

Geometry demonstrates:

1st. That the circumference of a circle is greater than the periphery of any inscribed polygon.

2nd. That the circumference is smaller than the periphery of any circumscribed polygon.

3d. That by continually doubling the number of sides of an inscribed polygon, its periphery becomes larger, and by continuing the operation will come nearer and nearer to the circumference of the circle.

4th. That by continually doubling the number of sides of a circumscribed polygon, its periphery becomes smaller, and also nearer to the circumference of the circle.

5th. That the circumference of the circle will be always between all inscribed and circumscribed polygons.

As now the sides and peripheries of all those polygons are easily found, we know the limits between which the length of the circumference of the circle must be situated, or in other words, knowing the periphery of a circumscribed polygon of a great number of sides, we know the limit which the number representing the relation between diameter and circumference cannot surpass, and knowing the periphery of an inscribed polygon of a great number of sides, we know the limit above which the number must be situated.*

The shortest way now to calculate the periphery of those polygons is to find a formula for the side of a polygon of double the number of sides of a given polygon, and also for the side of the circumscribed polygon of the same number of sides as the inscribed one, the latter being given.

Calling the side of the inscribed polygon s, and of the polygon of double the number of sides s', we easily obtain, taking the diameter of the circle $= 1$, the expression

$$s' = \sqrt{\tfrac{1}{2}\left[1 - \sqrt{(1-s^2)}\right]}$$

and calling S the side of the inscribed polygon of the same number as the inscribed one of s sides, we have

$$S = \frac{s}{\sqrt{(1-s^2)}}$$

Polygons of 6 Sides.

If now we commence with a polygon of six sides, then if the diameter of the circle being $= 1$, $s = \frac{1}{2}$ and

$$S = \frac{\frac{1}{2}}{\sqrt{(1-\frac{1}{4})}} = \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}$$

The periphery of the inscribed polygon of 6 sides is therefore $6 \times \frac{1}{2} = 3$, of the circumscribed $= 6 \times \frac{1}{3}\sqrt{3} = 2\sqrt{3}$. The length of the circumference of the circle is between these two numbers, and by doubling the sides we inclose it between narrower limits.

Polygons of 12 Sides.

So we will find the side of the inscribed polygon of 12 sides

$$s' = \sqrt{\tfrac{1}{2}\left[1-\sqrt{(1-s^2)}\right]} = \sqrt{\tfrac{1}{2}\left[1-\sqrt{(1-\tfrac{1}{4})}\right]} = \sqrt{\left(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{\tfrac{3}{4}}\right)}$$
$$= \sqrt{\left(\tfrac{1}{2}-\tfrac{1}{4}\sqrt{3}\right)} = \tfrac{1}{2}\sqrt{(2-\sqrt{3})}.$$

* As this essay is not intended for an elementary treatise on Geometry, we have no space to copy the demonstrations of these truths, or of the next following formulæ, they may be found in many of the larger works on Geometry, as LEGENDRE, LACROIX, and others.

and its periphery $= 12 \times \frac{1}{2}\sqrt{(2-\sqrt{3})} = 6 \times \sqrt{[2-\sqrt{3}]}$

The side of the circumscribed polygon of 12 sides is now found by placing in the formula

$S = \dfrac{s}{\sqrt{(1-s^2)}}$, for s the value $\frac{1}{2}\sqrt{(2-\sqrt{3})}$, and we obtain

$$S = \frac{\frac{1}{2}\sqrt{[2-\sqrt{3}]}}{\sqrt{[1-\frac{1}{4}(2-\sqrt{3})]}} = \frac{\sqrt{[2-\sqrt{3}]}}{\sqrt{[2+\sqrt{3}]}} = \frac{\sqrt{[2-\sqrt{3}]}\times\sqrt{[2+\sqrt{3}]}}{\sqrt{[2+\sqrt{3}]}\times\sqrt{[2+\sqrt{3}]}}$$

$$= \frac{\sqrt{[4-3]}}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}} = \frac{1\times[2-\sqrt{3}]}{[2+\sqrt{3}]\times[2-\sqrt{3}]} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

and its periphery $= 12 \times [2 - \sqrt{3}]$.

The periphery of the inscribed polygon of 12 sides is thus $6\times\sqrt{[2-\sqrt{3}]}$, and of the circumscribed $12\times[2-\sqrt{3}]$, and the circumference of the circle again between the two.

Polygons of more Sides.

This calculation may be shortened in the consecutive operations; we have, however, no room for the details, and will give only the results in the following table.

Number of Sides.	Side of Polygon; Inscribed	Side of Polygon; Circumscribed	Periphery of Polygon; Inscribed	Periphery of Polygon; Cicumscri'd
6	0.5	0.577350269etc.	3.0000000	3.4641016
12	0.258819045102	0.267949192 “	3.1058285	3.2153903
24	0.130526192220	0.1316525 “	3.1326286	3.1596600
48	0.065403129230	0.0655435 “	3.1393502	3.1460362
96	0.032719082821	0.0327190 “	3.1410319	3.1427146
192	0.016361731626	0.0163617 “	3.1414525	3.1418731
384	0.008181139604	0.00818113 “	3.1415576	3.1416630
768	0.004090604026	0.00409060 “	3.1415839	3.1416102
1536	0.002045306291	0.0020462 “	3.1415904	3.1415970
3072	0.001022653680	0.00102265 “	3.1415921	3.1415937
6144	0.000511326907	0.00051132 “	3.1415925	3.1415429
12288	0.000255663462	0.000255663463	3.1415926	3.1415927

This Table demonstrates how the peripheries of the inscribed and circumscribed polygons come nearer to each other by increasing the number of their sides; those of 12288 sides do not differ more than 1 ten millionth part of the diameter, and as the circumference of the circle must be between the two, the number 3.1415926 is certainly correct as far as the decimal fraction is written down, 3.1415927 being surely too great; let us take the mean between those two, namely, 3.14159265 which will be correct to within less than a hundred millionth part of the diameter, that is for a circle of which the diameter is equal to that of the whole earth, the circumference, valued after the above number, will be correct to within a single inch.

ARCHIMEDES is the first mathematician who found, 2000 years ago, an approximate value for this relation, or at least he is the first who found the limits between which its value is situated. Not possessing the use of algebriac formulæ* he only calculated the polygon of 96 sides, and found the limits between $3\frac{10}{70}$ and $3\frac{10}{71}$. He adopted the largest of these two numbers, because of its simplicity.

$$3\tfrac{10}{70} = 3\tfrac{1}{7} = \tfrac{22}{7} = 22 : 7$$

the last, being his celebrated ratio between the circumference and the diameter, gives the first $\frac{1}{497}$ too large.

A long time afterwards METIUS a Hollander, extended this calculation to polygons of about 1536 sides, and found the relation 3.1415929 which is correct to within three ten millionth part of the diameter. This number gives the beautiful ratio of 113 : 355, as it is so easily retained in memory, being the three first odd numbers, each repeated twice, 11,33,55, and then separated in two parts thus, 113 | 355.

VIETA went in his calculation about as far as we have done above, and ROMANUS extended it to 17 decimal figures. LUDOLP VAN CEULEN, also a Hollander, demonstrated in 1590 that

* Algebra was introduced only 200 years ago by DESCARTES.

if the diameter of a circle is represented by 1 followed by 35 ciphers, the circumference will be greater than

314159 265358 979323 846264 338327 950288

and smaller than

314159 265358 979323 846264 338327 950289

This number is called after the inventor the Ludolphian number, and is so near the truth that if we make a circle of which the radius is the nearest fixed star, we may calculate its circumference correctly to within a space much less than the thickness of a hair. Our imagination is at a loss to represent the smallness of such a fraction. His book on the properties of the circle has been published in the Dutch language in 1610, at *Van Ceulen's* death, and translated into the Latin in 1615, this being at that time the universal language of all scientific men. The above number is engraved on VAN CEULEN's tombstone, in the city of Leyden, Holland.

The occasion of all these calculations was to refute the theories of those who, without understanding much of Geometry, pretended to have discovered a particular method of finding the relation in question, but always came to a number larger than a certain circumscribed polygon, or smaller than a certain inscribed polygon.*

* Such kinds of persons are always so busy at work at this problem, (it may be considered a chronic decease of a fraction of society,) that the Academy of Sciences in Paris have several years ago found it necessary to adopt a resolution not to receive any more papers on that subject. Even here in New York a work appeared a few years ago pretending to demonstrate that the diameter being 6561 the circumference is 20612. Reducing this to our standard of the diameter =1, the circumference would be 3·1415914; less correct than the old ratio of METIUS, 113·355, and two millionth part larger than the circumscribed polygon of 3072 sides; however, as that author, to save his ratio, maintains that those inscribed polygons as soon as they have numerous sides, go inside the circle, or what is the same, that the circumference of the circle is outside of the circumscribed polygon, it would be wasting time and paper to say anything about such a book; it would do nobody any good, and surely not its author.

Later LAGNY a French Mathematician had the patience to calculate this number to 121 decimal figures, and lastly we have the famous number of 155 decimals, found in the library of Ratcliff, in Oxford, namely the diameter being = 1 the circumference is

3·14159	26535	89793	23846	26433
83279	50288	41971	69399	37510
58209	74944	59230	78164	06286
20899	86280	34825	34211	70679
82148	08651	32823	06647	09384
46095	50582	37172	53594	08128
4802	+ . . .	etc.		

We give this series of numbers only for the sake of curiosity, it is clear that for the most delicate calculations we have enough at 10 decimal figures.

At present, therefore, the relation of the diameter and circumference of a circle is considered to be a perfectly known quantity, at least much more accurately known than several other quantities which are daily used; for instance, nobody ever had, till the present day, the patience to calculate √ 2 or √ 3 to 155 decimal figures.

Algebra,* or rather higher arithmetic, teaches how to reduce the above ratio into a continued fraction, and from this to obtain simple approximate ratios in whole numbers; they are

Diameter	1	Circumference	3
"	7	"	22
"	106	"	133
"	113	"	355
"	33102	"	103993
"	33215	"	104348
	etc.		etc.

* We do not mean to say that algebra is higher arithmetic, but that the finding of such numerical ratios, strictly speaking, belongs to arithmetic, and not to algebra.

those numbers represent ratios alternately smaller and larger than the circumference.so is 1 : 3 too small.7 : 22 too large. 106 : 333 too small, 113 : 355, too large, etc. Of all numbers one may select those are the smallest possible for representing the nearest ratio of the the diameter to the circumference. So if we for instance adopt for the diameter a number between 113 and 33102 we cannot possibly find a whole number representing the ratio of the circumference which will be more correct than 113 : 355, unless we take 33102 : 103993.

In 1754 MONTUCLA published the history of the researches concerning the quadrature of the circle, adding this to his title : *A Book intended to make known the Real Discoveries concerning this Celebrated Problem, and to serve as a Preventative against new attempts at Solution.* So it is seen that for more than two centuries we have been troubled with quadrature hunters, and they are not dead yet.

LAMBERT demonstrated in 1761, and later LEGENDRE in his *Elemens de Geometrie* that the circumference is incommensurable with the diameter, from which demonstration it immediately follows that it is impossible to find two whole numbers representing their relation, and even if this was not so, and two such numbers could be found, it would be of little value at present after all we know now about it ; it would merely be an object of curiosity.

Let us now see if the relation of the diameter to the circumference can be represented by a single irrational number, or by a combination of irrational numbers ; if this was true, it would be possible to find a geometrical construction for the length of the circumference as irrational numbers and their combinations may easily be constructed by common geometry, when they are not above the second degree.

Giving the circumference of the circle of which the diameter is = 1, the customary Greek letter π, we have the following expression, found by J. BERNOUILLI when investigating the logarithms of the so called imaginary quantities:

$$\tfrac{1}{2}\pi = \frac{\log. \sqrt{-1}}{\sqrt{-1}}$$

WRONSKI in his *Introduction à la Philosophie des Mathematiques*, page 26, remarks truly about this expression that it contains logarithms, which are derivative functions, and that to get a theoretical expression for any quantity,—that is an expression which will declare the nature of the quantity to us—we can use only primitive functions, (such as arise from addition, multiplication, and evolution, or their converse operations,) he obtains the expression:

$$\tfrac{1}{2}\pi = \frac{\infty}{\sqrt{-1}} \left\{ (1 + \sqrt{-1})^{\frac{1}{\infty}} - (1 - \sqrt{-1})^{\frac{1}{\infty}} \right\}$$

in which really only primitive functions are found, and which reveals to us the nature of the celebrated number π; for since the equation is not of the 2nd., 3d., 4th., or any definite degree, but is of an infinite order, so it is demonstrated that the number π cannot be obtained by a finite construction, either Algebraic or Geometrical. Consequently the many constructions invented are only approximations, and impossible to demonstrate Geometrically.*

However, if we develope by the binomial formula of NEWTON the expressions

$$(1 + \sqrt{-1})^{\frac{1}{\infty}} \text{ and } (1 - \sqrt{-1})^{\frac{1}{\infty}}$$

we obtain the series

$$\pi = 4\,(-\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} + \tfrac{1}{17})\text{—etc.}$$

This series was found by LEIBNITZ by means of the differential

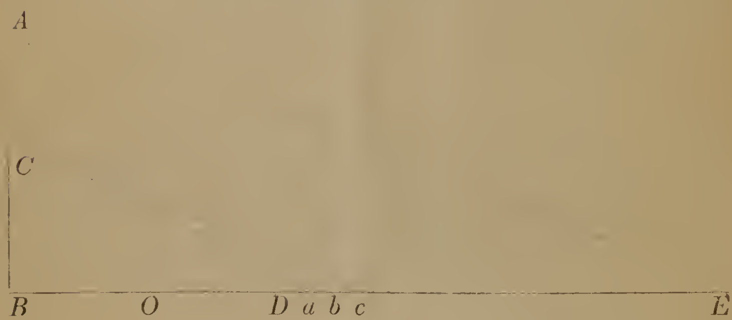
* The problem to make a straight line very nearly equal to the circumference of a circle of given diameter, has been solved in a great variety of ways; however of all those common Geometrical constructions we (either intentionally or accidentally) discovered or found elsewhere, there is one which

calculus. Below we will give such a calculation, different however fromthat which LEIBNITZ used to find the preceding one .

The convergence of this series however is not rapid enough for practical use, but it is easily changed into other series of such a nature that with their help it is less laborious to find 200 decimal figures than 20 with the method of polygons, described above.

is rather remarkable as possessing great simplicity, and being correct to within a half-millionth part of the diameter : for this reason we cannot refrain fromgiving it here. It will be found very useful for practical purposes.

Geometrical Construction for a very close approximation to the Circumference of the Circle.



Let BD be the diameter of a circle ; erect at its extremity B the perpendicular BC, and make it equal to the radius ; prolong BD, and make $Da = ab = bc$ = one-fifth of the radius : draw aC and cC, and make $BA = aC$. If now we draw from A the line AE parallel to cC, BE will be only one half millionth part smaller than the number π.

Demonstration :

As BD is $= 1$ $\quad Ba = 1.1$ $\quad$ and $Bc = 1.3$

$$Ca^2 = Ba^2 + BC^2 = \frac{146}{100} \quad \text{and } Ca = \frac{\sqrt{146}}{10}$$

Next we have $BC : Bc :: BA : BE$ or

$$\frac{1}{2} : \frac{13}{10} :: \frac{\sqrt{146}}{10} : Be \qquad \text{from which follows}$$

$$BE = \frac{13\sqrt{146}}{50} = 13\sqrt{0,0584} = \sqrt{9,8696} = 3,1415919 \text{ etc.}$$

We will now give the manner of operating to find the number π by means of the differential and integral calculus, applied to Goniometry; it may serve as an illustration how the preceeding and following remarkable series are found.

Let z be the arc of a circle, and x the tangent of this arc, or

$$x = \text{tang } z$$

for the radius $= 1$

All that we have to do is to find the value of z in this equation, because the whole problem is solved as soon as we know the value of an arc by its tangent, and if we could find a general expression for z in a function of x, the tangent of 45° being equal to the radius, and making $x = 1$ we shall have $\frac{1}{4}\pi = z$ and we will have the value of π.

To obtain this result let us observe that the differential of the arc is $=$ the differential of its tangent

$$dx = dz \quad \text{or} \quad dz = d \text{ tang } z \qquad (a)$$

but

$$d \text{ tang } z = d \left\{ \frac{\sin z}{\cos z} \right\} = \frac{\cos z \,.\, d\sin z - \sin z \,.\, d\cos z}{\cos^2 z}$$

the differential calculus teaches us that

$$d \sin z = \cos z \,.\, dz \text{ and } d \cos z = -\sin z \,.\, dz.$$

and substituting these values, we obtain

$$d \text{ tang } z = \frac{\cos^2 z \, dz + \sin^2 z \, dz}{\cos^2 z}$$

This last equation gives us, because $\cos^2 z + \sin^2 z = 1$,

$$dz = \cos^2 z. \, d \text{ tang } z \qquad (b)$$

In order to make the auxiliary quantity $\cos^2 z$ disappear, let us remember that

$$\sin z = \cos z \,.\, \text{tang } z$$

and consequently

$$\cos^2 z + \cos^2 z \,.\, \text{tang}^2 z = 1$$

from which follows

$$\cos^2 z = \frac{1}{1 + \text{tang}^2 z}$$

Substituting in (*b*) we have

$$dz = \frac{d \text{ tang } z}{1 + \text{tang}^2 z}$$

or, by substituting x for tang z

$$dz = \frac{dx}{1 + x^2} \qquad [c]$$

If now we take the Integral of the two members of this equation we have

$$z = \int \frac{dx}{1 + x^2} + C \qquad (d)$$

C being an arbitrary constant quantity, the value of which we will determine later on.

To find the arc z we have to integrate the expression

$$\frac{dx}{1 + x^2}$$

this is done in the following manner: we have

$$\frac{dx}{1 + x^2} = dx\,(1 + x^2)^{-1}$$

Developing the binomium $(1 + x^2)^{-1}$ by Newton's formula, and multiplying each member by dx, we obtain

$$\int \frac{dx}{1 + x^2} = \int \left\{ dx - x^2\,dx + x^4\,dx - x^6\,dx + x^8\,dx - \text{etc.} \right\}$$

taking the integral of each term, and observing that in general

$$\int x^m\,dx = \frac{x^{m+1}}{m + 1}$$

this expression becomes :

$$\int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^6}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \text{etc.}$$

and we have at last

$$z = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \text{etc.}$$

as regards the constant quantity spoken of above, it is = 0, because if we observe that if $z = 0$, we must have $x = 0$, and in this case the equation becomes $0 = 0 + C$, from which immediately follows $C = 0$.

This now is the series which gives the arc by the tangent; thus making $x = 1$, we must have $z = \frac{1}{4}\pi$, and obtain the remarkable expression also found by LEIBNITZ, but by a very different and far more laborious operation

$$\pi = 4\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.}\right)$$

The following expression is deduced from the new functions introduced in the science of numbers by VANDERMONDE and KRAMP.

$$\tfrac{1}{2}\pi = 2\left\{\left(\frac{1}{2}\right)^{\frac{1}{2}|-1}\right\}^2 \quad \text{or rather} \quad \tfrac{1}{2}\sqrt{\pi} = \left(\frac{1}{2}\right)^{\frac{1}{2}|-1}$$

Which expression teaches us also that the number π is not an irrational number of a certain power, but belongs to a higher order of quantity than the common irrational numbers.

The celebrated mathematician WALLIS found the following equation :

$$\tfrac{1}{2}\pi = \frac{2.2.4.4.6.6.8.8.10.10.12.12.\text{etc.}}{1.3.3.5.5.7.7.9.9.11.11.13.\ \text{etc.}}$$

This fraction possesses the peculiar property to give alternately values greater and smaller than half the circumference, by taking an even or odd number of terms. That means by reducing this fraction to a decimal one, we are obliged to take a certain number of terms, and to neglect the rest of them; so is $\frac{2}{1}$ too large, and $\frac{2.2}{1.3}$ too small. In the same way $\frac{2.2.6.4.6}{1.3.3.5.5}$ is too large, and $\frac{2.2.4.4.6.6}{1.3.3.5.5.6}$ too small, and in this way we obtain values by which half the diameter is brought continually within narrower limits.

In a way similar to that we followed p. 19 to find the series of LEIBNITZ, we may by means of the differential and integral calculus, easily find the following series:

$$\tfrac{1}{2}\pi = 1 + \frac{1}{2\times 3} + \frac{3}{2\times 4\times 5} + \frac{3\times 5}{2\times 4\times 6\times 7} + \frac{3\times 5\times 7}{2\times 4\times 6\times 8\times 9} \text{ etc.}$$

Again by the same calculus we find the length of an arc corresponding to a given sine x:

$$\text{arc}\,(\sin x) = x + \frac{1\,x^3}{2\times 3} + \frac{1\times 3\times x^5}{2\times 4\times 5} + \frac{1\times 3\times 5\times x^7}{2\times 4\times 6\times 7} + \text{etc.}$$

If now we take into consideration that the arc of a sixth part of the circumference is equal to the radius, and therefore the sine of the twelfth part of the circumference is equal to half the radius, we will have

$x = \frac{1}{2}$ and arc $(\sin x) =$ arc $(\sin \frac{1}{2}) = \frac{1}{6}\pi$ and therefore

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1\times 1}{2\times 3\times 2^3} + \frac{1\times 3\times 1}{2\times 4\times 5\times 2^5} + \frac{1\times 3\times 5\times 1}{2\times 4\times 6\times 7\times 2^7} \text{ etc.}$$

which converges so rapidly that by taking only 10 terms we obtain

$$\pi = 6 \times (0{\cdot}52359877) = 3.14159262$$

correct for the true value of π to the eighth decimal place.

There exists still another class of expressions for the number π, namely the form of continued fractions.* We give here one, by the invention of which BROUNKER has become celebrated among mathematicians.

$$\tfrac{1}{4}\pi = \cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\text{etc.}}}}}}}$$

in which the numerators are the squares of the odd numbers 1, 3, 5, 9, etc.

This expression, however, is nothing but a transformation of the series of LEIBNITZ, page 16, and which we found by an easy method; (page 19) it is just as little convergent as LEIBNITZ's series, which means that a certain number of its terms gives the same value as a similar number of terms of LEIBNITZ's series, those values are:

for 1 term	0 6666666, etc.
" 2 "	0·8733333 "
" 3 "	0·7238094 "
" 4 "	0·8347205 "
" 5 "	0·7440115 "
" 6 "	0·8209347 "

By which calculation it is seen that 7 terms are not sufficient to give the correct value of the first figure in the decimal fraction.

* In the Appendix will be found a few words to illustrate those fractions.

But we may transform the series of LEIBNITZ in another continued fraction, which was found by DE MONTFERRIER:

$$\frac{1}{4}\pi = \cfrac{1}{1 + \cfrac{\frac{1}{1.3}}{1 + \cfrac{\frac{4}{3.5}}{1 + \cfrac{\frac{9}{5.7}}{1 + \cfrac{\frac{16}{7.9}}{1 + \text{etc.}}}}}}$$

The law of progress in this fraction is easy to understand: the numerators of the special fractions are the squares of the successive numbers 1.2.3.4.5, etc., and the denominators are the products of the successive odd numbers taken two each time 1.3, 3.5, 5.7, etc.

But we may again bring this fraction to another form, namely:

$$\frac{1}{4}\pi = \cfrac{1}{1+\cfrac{1}{3+\cfrac{4}{5+\cfrac{9}{7+\cfrac{16}{9+\cfrac{25}{11 + \text{etc.}}}}}}}$$

This fraction has stronger convergence than any of the others, as the following calculation indicates:

Value of 1 term,	1
" " 2 "	0·75
" " 3 "	0 7915666 etc.
" " 4 "	0·7843120 "
" " 5 "	0·7855855 "
" " 6 "	0 7653657 "
" " 7 "	0·7654037 "

By only these seven terms we obtain for a quarter of the circumference; 0·7854037 that is, for the whole circumference, $\pi = 4 \times 0{\cdot}7854037 = 3{\cdot}1416148$, which varies a little over a two hundred thousandth part from the truth; by taking 8 terms we are correct to within one hundred millionth part, and this proves that we now possess easier means to obtain the correct ratio of the diameter to the circumference, than all hunters after the quadrature will ever be able to find.

PIOCHE of Metz found the following expression, which proctures us an easy construction, similar to that given on page 18.

$$\pi \text{ very near } = \tfrac{1}{2}(3+\tfrac{1}{5}+\tfrac{3}{4}+\tfrac{2}{3}\sqrt{[1+9]}\,) = 3.1415925 \text{ etc.}$$

EULER occupied himself extensively with finding peculiar algebraic expressions for the celebrated number π. We will give a few more of his series:

$$\pi = \tfrac{2}{3}\,[\,1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \text{ etc. }]$$

or

$$\pi = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} - \tfrac{1}{11} + \text{etc.}$$

in the last series all binary and ternary numbers have the + sign, also the primary numbers of the form $4\,m+1$; however those of the form $4\,m-1$ have the − sign.
and finally we give one more:

$$\pi = 2\sqrt{2}\left\{\frac{3}{4}, \frac{5}{4}, \frac{7}{6}, \frac{11}{12}, \frac{13}{12}, \frac{17}{18}, \frac{19}{20}, \frac{23}{22}, \text{etc.}\right\}$$

We have no place for more of such expressions, it would make this essay too large to give them all ; those of our readers who desire to know them in all particulars, will obtain perfect satisfaction by his work:— *Introductio in Analysin Infinitorum* a Latin work, of which translations exist in French and German, and which is not as generally known among Mathematicians as it deserves. It is found in most good Libraries, for instance in the Astor Library, New York, which, by the way, is as well provided with standard works in the Mathematical department of science as any library in the civilized world.*

In conclusion we give a few practical expresssions relative to the Quadrature:

Diamcter $= 1$. *Circumference* or $\pi = 3.1415926$ etc. see page 15.

Surface $= \frac{1}{4}D^2\pi = 0.78539816339744830961$ etc. square units.

Side of equivalent Square $= \frac{1}{2}D\sqrt{\pi} = 0.889226925452758$ etc.

Surface of Sphere $= D^2\pi = 3.14159265358 9$ etc. square units.

Volume " " $= \frac{1}{6}D^3\pi = 0.52359877559829$ &c. cubic units.

Lengt of Arc of $30^\circ = \frac{1}{12}\pi = 0.26179938779914943653855$ etc.

" " $1^\circ = \frac{1}{360}\pi = 0.008726646259971647884618$ etc.

$\frac{1}{\pi} = \pi^{-1} = 0.3183098861837906715377675267450$ etc.

$\pi^2 = 9.869604401089358618834490999876$ etc.

$\sqrt{\pi} = 1.7724538509055160272981674833341$ etc.

$\frac{1}{\pi^2} = \pi^{-2} = 0.101321183642337771443879463209$ etc.

* We are sorry that we cannot say the same of the Medical department of this Library. The Physicians of New York and its vicinity will have to offer a petition to the Trustees of the Astor Library praying to be as liberally provided for as the Theologians.

APPENDIX.

Persons not very familiar with Mathematics are sometimes surprised at continued fractions, and series without end, not thinking that in arithmetic such cases are of daily occurence, for instance, in reducing common to decimal fractions; if namely, the denominator of the common fraction is not any power of 2 or 5, or a product of such, the decimal fraction obtained for the value of the common fraction will be infinite. for example

$\frac{1}{3} = 0{\cdot}33333$, etc. $\frac{1}{6} = 0{\cdot}166666$, etc.

$\frac{1}{7} = 0{\cdot}142857142857$, etc. $\frac{1}{9} = 0{\cdot}1111111$, etc.

$\frac{1}{11} = 0{\cdot}090909090$, etc. $\frac{1}{13} = 0{\cdot}076923076923076923$0 etc.

It is seen that if the same number is not repeated, such is the case with a series of two, three, or more numbers; we may reduce every periodically repeating decimal fraction to a common fraction in the following way:

Suppose we have the continued and periodical decimal fraction 0·090909, etc., we take this fraction 100 times,

	100 times the fraction	= 9.090909, etc.
and substract	1 " " "	= 0,090909, etc.
	99 " " "	= 9

dividing this equation by 99 we have

$$1 \text{ time the fraction} = \frac{9}{99} = \frac{1}{11}$$

and the same for all other continued decimal fractions.

In calculating roots which are irrational numbers, for instance $\sqrt{}$ 3, 5, 6, 7, etc, we obtain also continued decimal fractions, in which, however, no repetition of the same series of numbers is observed; these decimal fractions, therefore, are not periodical.

and cannot possibly be, because, as we have seen above, every periodical decimal fraction can be converted into a common fraction, which in this case, would represent the root of a whole number. However, as the squares, cubes, etc., of a common fraction must necessarily be a fraction also, it is clear that a common fraction cannot be the root of a whole number, and consequently a continuous decimal fraction, representing a root approximately, can never repeat, but must change the order of the figures infinitely.

However, there are expressions which represent those roots in a periodical or repeating form, namely, the continued fractions, in which the denominators will return in the same order; so we have

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}}} \qquad \sqrt{3} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \&c}}}}}$$

Reducing such a fraction we obtain approximations to irrational roots, so is $\sqrt{2} = 1.4142135623730950488016887242097$ etc.

$$\sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \text{etc.}}}}} \qquad \sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \text{etc.}}}}}$$

$$\sqrt{10} = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{\text{etc.}}}}}} \qquad \sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{\text{etc.}}}}}}$$

$$\sqrt{13} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{\quad + \cfrac{1}{1 + \text{etc.}}}}}}}} \qquad \text{or} \qquad \sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \text{etc.}}}}}}$$

There is a simple way of finding such fractions for the irrational roots ; let us adopt a general repeating continued fraction, and make its unknown value $= x$, or let us adopt

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \text{etc.}}}}} \qquad [e]$$

this fraction being supposed infinite, there is no difference in value if we commence it from the first or from the second numerator, and therefore we must have

$$x = \frac{1}{a + x}$$

the last x being substituted for the rest of the fraction.

From this equation follows

$$a\,x + x^2 = 1$$

an equation of the second degree, giving for x

$$x = -\frac{a}{2} \pm \frac{1}{2}\sqrt{\left(a^2 + 4\right)}$$

Let us now for an example adopt $a = 2$. we will have

$$x = -1 \pm \tfrac{1}{2}\sqrt{8} = -1 \pm \sqrt{2}, \text{ or }, \sqrt{2} = 1 + x,$$

and now substituting the value of x from equation [e]

$$\sqrt{2} = 1 + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \text{etc.}}}}$$

in which a is to be taken $= 2$ to produce the expression for $\sqrt{2}$ given pag 28.

For $a = 6$ we have $x = -3 \pm \sqrt{10}$ and $\sqrt{10} = 3 + x$.

$$\sqrt{10} = 3 + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \text{etc.}}}}$$

in which a is to be made $= 6$ to find the expression for $\sqrt{10}$, see pag. 28.

Let us lastly, for a second example adopt a general continued fraction repeating two figures alternately

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \text{etc.}}}}}$$

If here we substitute x for the rest of the fraction after the second denominator, we will have

$$x = \cfrac{1}{a + \cfrac{1}{b + x}}$$

from which equation follows

$$x = \frac{b + x}{ba + ax + 1} \quad \text{or, } ax^2 + abx = b$$

an equation of the second degree, giving in its solution

$$x = -\frac{b}{2} \pm \sqrt{\left(\frac{ab^2 + 4b}{4a}\right)}$$

consequently
$$\sqrt{\left(\frac{ab^2 + 4b}{4a}\right)} = \frac{b}{2} + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \text{etc.}}}}}$$

And adopting different arbitrary values for a and b we obtain expressions for different roots; so, for instance, taking $a = 2$, and $b = 4$, we have

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \text{etc.}}}}}$$

At the same time the above illustrations show how every periodical continued fraction may be reduced to an irrational expression.

The continued fractions which we have given p. 24 for the circumference of the circle are not periodical; if they were, they could be reduced to an irrational expression, and as those which are not periodical, can not be reduced to such, it is clear that there exists no irrational expression for the number π, and that the ratio between the diameter and circumference is of a higher order than the common irrational quantities, and compares with them as the irrational quantities compare with common fractions.

Irrational quantities of the 2nd. degree may be constructed by common geometry; those of the 3rd. and 4th. degrees by conic sections; of the 5th and higher degrees by higher mathematics. Quantities of an infinite order however, cannot be constructed geometrically, and consequently there can not possibly exist a mathematical construction for the perfect value of π the circumference of the circle. See page 17.

And so we have finished our task, which was to demonstrate that all farther investigation of the Quadrature of the Circle is useless. It is a problem at this time solved in many ways, perhaps even in all possible ways, and as all the different methods of solution give absolutely the same result, if we follow either common Geometry using the method of polygons, or Goniometry applying the Differential Calculus, or any other method of which only a few have been treated by us, we must conclude from this alone, that the result obtained must be correct, even if each method did not carry with it, its absolute mathematical demonstration.

Ergo: Those persons who do not know that the problem of the Quadrature of the Circle has been solved long ago, are more than a century behind the age, and do not understand anything about Modern Mathematics;

Those who think that it can not be solved at all, never received Elementary Instruction in Mathematics, in a thorough manner;

and finally, Those who in our age occupy themselves with a research after a ratio in whole numbers, or a Geometrical Construction, (a thing we have shown to be as impossible as to find a whole number for $\sqrt{3}$,) belong to that unfortunate class of persons who deserve our pity.

APPENDIX

TO THE SECOND EDITION OF THE

QUADRATURE

OF THE

CIRCLE

BY

P. H. VanderWeyde, M. D.

NEW YORK:
D. APPLETON & CO.,
1862.

ERRATA.

Page 14, Note, 5th line from the bottom, instead of "inscribed polygons" read "circumscribed polygons."

APPENDIX TO THE SECOND EDITION

OF THE

QUADRATURE OF THE CIRCLE.

Since the publication of his pamphlet (only one year ago) not less than five persons have applied to the Author, communicating to him that they had found the *Quadrature of the Circle*, and six others who believed they had found *Perpetual Motion*. His pamphlet has been given in answer to the Circle Squarers, with what result he is unable to determine.

Prof. Henry of the Smithsonian Institute and Prof. Bache of the Coast Survey in Washington, informed the Author that both Institutions are sometimes bored with similar communications from persons in different distant parts of the Continent. Some of them entertain the notion that a large premium for the squaring of the circle has been promised by one or more of the European Governments; while to the contrary the Academy of Paris, already in 1775, and soon after the Royal Society of London, declined to examine in future any paper pretending to solve the *Quadrature of the Circle, Trisection of an Angle, Duplication of the Cube, or Perpetual Motion*, hoping therewith to discourage these and other similar researches.

Notwithstanding this, several circle squarers promised large premiums to all who could convince them of error; those premiums were claimed by many, often before judicial Courts; the circle squarers were either sentenced to pay, or in most cases the offer was declared void; a very just sentence, taking into account the mental delusion under which all circle squarers labor.

It is stated, and appears to be a fact, that of all the circle squarers only one was ever fully convinced of his error; his name deserves to be preserved, he was an English Jesuit Priest named *Richard White.*

One of the periodical class of Books spoken of, p. 14 above, appeared last year in England, written by a certain *Mr. James Smith*: who attempts to prove that the circumference is exactly $3\frac{1}{8}$ diameters, and denies that it lies between the inscribed and circumscribed polygons.

It is surprising that it was thought worth while in England to take notice of the book; but this was the case. The *British Association* examined into the subject, an able Mathematician entered in correspondence with Mr. Smith, vainly thinking that he could persuade him of his error; the *London Athenæum* devoted more than four columns to it; and *Sir Rowan Hamilton* published a geometrical proof that the perimeter of an inscribed polygon of 20 sides was larger than $3\frac{1}{8}$ diameters, and still more of course the circumference $>3\frac{1}{8}$ diameters. This proof is a part of our calculation given on page 12.

The *Athenæum* asks with good reason why the question of *crossing the square* has not been as celebrated as that of *squaring the circle?* Both relations being irrational quantities cannot be expressed by whole numbers (see p. 32,) but the first was never attempted because Euclid already demonstrated the nature of that question, while that of the last was not demonstrated completely until within this century.

In addition to the results of calculations mentioned page 15, we must remark that the 155 figures given there as the relation between diameter and circumference have been proved correct except the last two numbers. Dr. Rutherford of Woolwich presented a calculation of 200 figures to the Royal Society, London, of which however all those added to the 155 mentioned above were wrong; perhaps he was confident that nobody would take the pains to persuade him of error; this was however done by Dr.

Clausen of Dorpat, who calculated 250 decimals, and Mr. Shanks of Durham, 315 decimals. This stirred Mr. Rutherford up, and he tried to find errors, but found the figures all correct, and he extended them to 350 decimals. Mr. Shanks appears to have become jealous, and carried them to 527 decimals. Mr. Rutherford, wishing to ascertain if they were correct, found them so to 411 decimals, and then gave it up. Mr. Shanks did not give it up, and went again to calculating till he had obtained 607 decimals, and published the result of his calculations in the *Contributions to Mathematics*, London, 1853.

This curious decimal fraction, representing the relation of the diameter to the circumference of the circle ALMOST CORRECT, is the following:

π =	3·14159	26535	89793	23846	26433
	83279	50288	41971	69399	37510
	58209	74944	59230	78164	06286
	20899	86280	34825	34211	70679
	82148	08651	32823	06647	09384
	46095	50582	23172	53594	08128
	48111	74502	84102	70193	85211
	05559	64462	29489	54930	38196
	44288	10975	66593	34461	28475
	64823	37867	83165	27120	19091
	45648	56692	34603	48610	49432
	66482	13393	60726	02491	41273
	72458	70066	06315	58817	48815
	20920	96282	92540	91715	36436
	78925	90360	01133	05305	48820
	46652	13841	46951	94151	16094
	33057	27036	57595	91953	09218
	61173	81932	61179	31051	18548
	07446	23798	34749	56735	18857
	52724	89122	79381	83011	94912
	98336	73362	44193	66430	86021
	39501	60924	48077	23094	36285
	53096	62027	55693	97986	95022
	24749	96206	07497	03041	23669
	29133	32± etc.			

Those 607 decimals of course give to every one the persuasion that an exagerated accuracy has been obtained ; the strongest imagination however cannot possibly realise its extent, as will be perceived from the following account which we quote from Knight's Cyclopedia.

"Say that the blood-globule of one of our animalcules is a millionth of an inch in diameter. Fashion in thought a globe like our own, but so much larger that our globe is but a blood-globule in one of its animalcules: never mind the microscope which shows the creature being rather a bulky instrument. Call this the first globe *above* us Let the first globe above us be but a blood-globule, as to size, in the animalcule of a still larger globe, which call the second globe above us. Go on in this way to the twentieth globe above us. Now go down just as far on the other side Let the blood-globule with which we started be a globe peopled with animals like ours, but rather smaller: and call this the first globe below us. Take a blood-globule out of this globe, people it, and call it the second globe below us: and so on to the twentieth globe below us. This is a fine stretch of progression both ways Now give the giant of the twentieth globe *above* us the 607 decimal places, and, when he has measured the diameter of his globe with accuracy worthy of his size, let him calculate the circumference of his equator with the help of the 607 decimals. Bring the little philosopher from the twentieth globe *below* us with his very best microscope, and set him to see the small error which the giant must make. He will not succeed, unless his microscopes be much better for his size than ours are for ours.

Now it must be remembered by any one who would laugh at the closeness of the approximation, that the mathematician generally goes *nearer ;* in fact his theorems have usually no error at all. The very person who is bewildered by the preceding description may easily forget that if there were *no error at all*, the Lilliputian of the millionth globe below us could not find a flaw in the Brobdingnagian of the millionth globe above. The three angles of a triangle are *absolutely* equal to two right angles ; no stretch of progression will detect *any* error.

The London Athenæum contains the following:

We will now, for our non-calculating reader, put the matter in another way. We see that a circle-squarer can advance, with the utmost confidence, the assertion that when the diameter is 1,000, the circumference is accurately 3125: the mathematiciam declaring that it is a trifle more than 3141.5. If the squarer be right, the mathematician has blundered by about a 200th part of the whole: or has not kept his accounts right by about 10*s*. in every 100*l*. Of course if he set out with such an error he will accumulate blunder upon blunder. Now, if there be a process in which close knowledge of the circle is requisite, it is the prediction of the moon's place—say, as to time of passing the meridian at Greenwich—on a given day. We cannot give the least idea of the complication of details: but common sense will tell us that if a mathematician cannot find his way round the circle without a relative error four times as big as a stockbroker's commission, he must needs be dreadfully out in his attempt to predict the time of passage of the moon. Now, what is the fact? His error is less than a second of time, and the moon takes 27 days odd to revolve. That is to say, setting out with 10*s*. in 100*l*. of error in his circumference, he gets within the fifth part of a farthing in 100*l*. in predicting the moon's transit. Now we cannot think that the respect in which mathematical science is held is great enough—though we find it not small—to make this go down. That respect is founded upon a notion that right ends are got by right means: it will hardly be credited that the truth can be got to farthings out of data which are wrong by shillings. Even the celebrated Hamilton, who held that in mathematics there was no way of going wrong, was fully impressed with the belief that this was because error was avoided from the beginning. He never went so far as to say that a mathematician who begins wrong must end right somehow."

When several centuries ago the problem was first attempted, the circle stood alone among the curves, and the remarkable distinction between rectilinear figures and the only curve then considered, could not but excite curiosity. This state of things is now changed; the circle is only one curve among an infinitely great number, many of which have been investigated. Consequently, according to the present state of science, the problem analogous with squaring the circle 1000 years ago, is now; ***Given any curve, to find its area.***

If therefore the ingenuity, guided by love for investigation, desires a field for exertion, why not leave the circle alone, which has been exhausted, and let it spend its force upon the other curves, among many of which the field for further investigation is unexplored For one point that strikes the lover of investigation in the quadrature of the circle, there are hundreds in other subjects. which alternately puzzle, surprise, and delight the mathematician. Besides, as the quadrature of the circle was once in the hands of Wallis, Euler, Newton, etc. a road to very different striking results not looked for, so the efforts of the inquisitive in the new field of our day, may end in the promotion of other parts of mathematical science, if only begun with the right preparation. The binomial theorem was due to the learned attempts of *Wallis* upon the quadrature, and we may perhaps hope for similar successes, if those who now labor without results, would only furnish themselves with the knowledge of what has been discovered by others, before they set up for discoveries themselves.

All men who have discovered anything in mathematics were learned men, fully acquainted with the subject they were intended to advance; they had all shown their power over what *was known*, before they presented themselves as the promulgators of what *was not known*. All squarers of the circle therefore may be asked proof of their acquaintance with the previous discoveries of the subject; any one who does this will always meet with some attention. But all circle squarers are as ignorant of the past as the future will be of them.